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On the QCD coupling behavior in the infrared region

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The summary of nonperturbative results for the QCD invariant coupling $\bar{\alpha}_s$ obtained by lattice simulations for functional integral and by solution of approximate Dyson–Schwinger equations reveals a puzzling variety of IR behavior of $\bar{\alpha}_s(Q^2)$ even on a qualitative level. This, in turn, rises a question of correspondence between the results obtained so far by different groups.

We analyze this issue in terms of mass-dependent coupling constant transformations and conclude that the question of the IR behavior of effective QCD coupling and of propagators is not a well-defined one and needs to be more specified.

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1 Introduction

This paper is devoted to the issue of QCD invariant (effective) coupling function $\bar{\alpha}$ behavior in the infrared (IR) region.

The notion of invariant coupling is the central one in current practice of quantum-field applications of the renormalization group (RG) method. In QFT, the very existence of RG is generically connected with Dyson's finite renormalization transformations. The invariant coupling (IC) — or invariant charge — has been introduced (see pioneer papers [1, 2, 3], as well as [4]) via a product of finite Dyson's renormalization factors z_i , the product that is invariant with respect to the mentioned transformations, without any reference to weak coupling or ultraviolet (UV) limits.

By expressing z_i via values of renormalized QFT functions one usually obtains an expression for IC $\bar{\alpha}(Q^2)$ as a function of one argument $Q = \sqrt{Q^2} = \sqrt{\mathbf{Q}^2 - Q_0^2}$ — momentum transfer (or reference momentum) — in terms of a product of the coupling constant, and the vertex and propagator Lorentz-invariant amplitudes taken in the momentum representation¹ in some particular renormalization scheme.

Such definitions have been introduced and then used in the perturbation case for the UV and IR asymptotics in the mid-50s. In current practice, $\bar{\alpha}(Q^2)$ is usually employed in the UV limit only. The massless minimal subtraction renormalization scheme $\overline{\text{MS}}$ turns out to be the most convenient one and, hence, more popular in the UV analysis.

Meanwhile, the mentioned definitions are valid in a more general, mass dependent, case — see refs.[1, 3, 4, 5]. In the gauge QFT models containing several vertices with the same coupling constant this mass dependence, in turn, yields a specific *vertex dependence*, *i.e.*, the dependence on a particular dressed vertex chosen for the IC defining. In QCD, for instance, one can use three-gluon, gluon-ghost, four-gluon and various gluon-quark vertices. In the massless $\overline{\text{MS}}$ scheme, all ICs constructed with the use of different vertices are the same. However, in the massive case, they could differ essentially [6] in the $Q \lesssim m$ region.

Here, the comment on the difference between massless and mass-dependent subtraction schemes and related RG solutions is in order. For example, there exist two variants of the $\overline{\text{MS}}$ scheme. In the common one, for the QFT model with one coupling constant g , with the help of massless counter-terms one constructs a beta-function $\beta(g)$, the function of one argument. The RG equation solving for IC results in expression $\bar{g} = f(\ell, g)$ depending on the logarithm $\ell = \ln(Q^2/\mu^2)$ and describing the UV asymptotic behavior. Meanwhile, in a more general formalism, the RG generators (beta and gamma functions) are defined with the help of the $\overline{\text{MS}}$ renormalized, mass-dependent approximate expressions. They are functions of two dimensionless arguments Q^2/m^2 and g . The RG equations solving produces expressions with the additional “massive” argument m^2/μ^2 . The general expression for IC is now of the form $\bar{g}(Q^2/\mu^2, m^2/\mu^2, g)$. The mass dependence is essential in the IR domain. This algorithm was used in QED[2, 7] for describing the IR singularity of the electron propagator.

This mass dependence, in our opinion, reveals itself in the current analysis of the IR properties in QCD, in particular, at lattice calculation of the functional integral and in solving the truncated Dyson-Schwinger equations (DSE). Various groups of researchers use different definitions of coupling constants and IC in the IR region. For instance, in the

¹See, *e.g.*, , eqs.(1), (2) and (3) below.

IC defining the Tübingen group uses[8] the gluon–ghost vertex, the Paris group [9, 10] — three–gluon one, while the “Transoceanic team” [11, 12] — the gluon–quark vertex.

Section 2 contains a short overview of different groups’ results for $\bar{\alpha}_s$ obtained by solution of truncated DSE as well as by lattice simulations in the IR region. This summary reveals a rather wide variety of IR behaviors.

In Section 3, we remind the origin of the IC $\bar{\alpha}_s(Q^2)$ notion and note that it is free of any relation with the weak coupling or UV massless limits. Then, we consider the item of nonuniqueness of the $\bar{\alpha}_s(Q^2)$ definition, and the related question of coupling constant and IC transformations. We discuss this issue in terms of coupling transformations similar to the renormalization–scheme ones in the weak coupling limit, but, generally, more involved and mass dependent. Here, the important feature is *the vertex dependence* that is typical of QFT models with gauge symmetry.

Further on, in Section 4, we return to the item of correspondence between different IR behavior of $\bar{\alpha}_s(Q^2)$. In particular, we give few examples of transformations that are singular in the IR region and, in some cases, results in the $\bar{\alpha}_s(Q^2)$ behavior in a similar way to the ones discussed in Section 2.

Our conclusion is that the question of the IR behavior of the effective QCD coupling $\bar{\alpha}_s(Q^2)$ (and of propagators) is not a well–defined one. The results of different groups, formulated in the momentum representation, should not be compared directly with each other. Calculation of hadronic characteristics remains as a reasonable criterion for comparison of different lattice simulation schemes.

2 Nonperturbative results in the IR region

Here, we give a short overview of nonperturbative results obtained by different groups in calculating the effective coupling $\bar{\alpha}_s(Q^2)$ behavior in the IR region by numerical simulation on lattice and by approximate solution of the truncated Dyson–Schwinger equations (DSEs).

Each of these groups uses its own definition for $\bar{\alpha}_s(Q^2)$, compatible with the asymptotic freedom regime in the UV region, and obtains its own result in the IR domain.

2.1 Tübingen: gluons and ghosts

We start with a description of recent results obtained by the Tübingen group. This group activity (for a review see paper [8]) proceeds in parallel along two lines: solving the truncated Dyson–Schwinger equations and the lattice simulation of Euclidean 4-dimensional functional integral for quantum gluodynamics. Thus, this group limits itself to the gluon–ghost sector of QCD that turns out to be convenient for the DSEs truncation in the Landau gauge. Here, the gluon–ghost vertex renormalization function Γ due to the Ward–Slavnov identities can be represented via analogous functions of the gluon Z and ghost G propagators. At the same time, in this gauge, Γ drops out from the invariant coupling defined as

$$\bar{\alpha}_{Tu}(Q^2) = \alpha_s Z(Q^2) G^2(Q^2). \quad (1)$$

Due to this, only two propagators enter into the properly truncated DSEs.

Lattice simulations for propagators reveal [13, 14] a rather specific IR behavior. The gluon function $Z(Q^2)$ — the transverse amplitude of the gluon propagator — passes through maximum at $Q = \sqrt{Q^2} \simeq 1$ GeV and then goes to zero approximately as Q , the ghost one $G(Q^2)$ is monotonous and has a power IR singularity close to $Q^{-1/2}$, while the product (1) tends to a finite value. This picture is supported by

DSEs solving that yields the power IR behavior for the propagators

$$G(Q^2) \simeq (Q)^{-\kappa}; \quad Z(Q^2) \simeq (Q^2)^\kappa \quad \text{with} \quad \kappa = 0.595$$

and finite limiting value for the invariant coupling

$$\bar{\alpha}_{Tu}(0) \simeq 2.97 (= 8.92/N_c).$$

2.2 Paris group, 3– gluon vertex

Here, the invariant QCD coupling function is constructed on the basis of the three–gluon vertex with the nonsymmetric MOM subtraction in the Landau gauge for the QCD model with **two quark flavors**. In this particular MOM scheme the invariant coupling

$$\bar{\alpha}_P(Q^2) = \alpha_s \tilde{\Gamma}^2(Q^2) Z^3(Q^2) \quad ; \quad \tilde{\Gamma}(Q^2) \equiv \Gamma_{3gl}(Q^2, 0, Q^2), \quad (2)$$

according to lattice simulations, obeys a very peculiar behavior in the IR region. It passes through the maximum at $Q \simeq 1 - 1.5$ GeV and then quickly approaches zero at $Q \simeq 0.5$ GeV – see Fig.3 from [9].

The data here are not sufficient to define the IR limiting regime. Nevertheless, in our opinion, the power decreasing $\simeq (Q^2)^\nu$; $\nu \gtrsim 2$ is not excluded.

2.3 “Transoceanic team”, gluon–quark vertex

This team² considers (for a recent review see [12]) the QCD model with **two massive quarks**. It defines its invariant coupling \bar{g}_{TO} on the basis of the gluon–quark vertex in the particular MOM scheme with gluon momentum equal to zero. It reads

$$\bar{g}_{TO}(Q^2) = \alpha_s \Gamma_{TO}(Q^2) Z^{1/2}(Q^2) S(Q^2) \quad ; \quad \Gamma_{TO}(Q^2) \equiv \Gamma_{q-gl}(0; Q^2, Q^2), \quad (3)$$

with S , the quark propagator amplitude. The results for \bar{g}_{TO} obtained by lattice simulation, qualitatively, are close to the “Paris group” ones – see Fig. 4 in paper [11] or Figs. 8 and 9 in [12]. However, the IR asymptotics of IC, evidently, has a different power

$$\bar{g}(Q) \simeq (Q^2)^{\mu/2}; \quad \bar{\alpha} \simeq (Q^2)^\mu; \quad \mu \lesssim 1.$$

²The group includes the authors from various centers from Australia, Great Britain and continental Europe.

2.4 “ALPHA” — The Schrödinger functional

The “ALPHA” group considers the QCD model with **two massless flavors**. It uses the *Schrödinger functional* (SF) defined in the Euclidean space–time manifold in a specific way: all three space dimensions are subject to periodic boundary conditions, while the “time” one is singled out — the gauge field values on the “upper” and “bottom” edges differ by some phase factor $\exp(-\eta)$. Then, the renormalized coupling is defined via the derivative $\Gamma' = \partial\Gamma/\partial\eta$ of the leading part of the effective action

$$\Gamma = \alpha^{-1}\Gamma_0 + \Gamma_1 + \alpha_s\Gamma_2 + \dots$$

as (cf. eq.(2.43) in ref.[15] and (8.3) in ref.[16]) a function *in the coordinate representation*

$$\bar{\alpha}_{\text{SF}}(L) = \Gamma'_0/\Gamma' \quad (4)$$

with L being the spatial size of the above–mentioned manifold.

To follow the $\bar{\alpha}_{\text{SF}}(L)$ evolution, a special trick was used [16]. The “step scaling function”

$$\sigma(\bar{\alpha}_{\text{SF}}(L)) = \bar{\alpha}_{\text{SF}}(2L)$$

has been introduced. For explicit implementation of σ , the beta–function is necessary. On the other hand, numerically, this function can be defined from lattice simulations by comparing the results for lattice L with $2L$ — see Fig.16 in ref.[16]. Meanwhile, fresh results [17] reveal the steep rise of the SF running coupling in the region $\bar{\alpha}_{\text{SF}} \simeq 1$ and $L \rightarrow \infty$. Here, the analytic fit for the numerically calculated behavior of $\bar{\alpha}_{\text{SF}}$ has an exponential form

$$\bar{\alpha}_{\text{SF}}(L) \simeq e^{mL} \quad \text{with} \quad m \simeq 2.3/L_{\text{max}} \quad (5)$$

and L_{max} , a reference point in the region of sufficiently weak coupling, indirectly defined via the condition $\bar{\alpha}_{\text{SF}}(L_{\text{max}}) = 0.275$. For a physical discussion of the IR (i.e., at $Q^2 \rightarrow 0$) QCD behavior, in the papers of ALPHA group the “usual quantum–mechanical correspondence” $L = 1/Q$ is used.

3 Mass–dependent coupling function

Renormalization schemes with scale parameter μ coinciding with subtraction momentum — the so–called MOM schemes — are singled out from a formal point of view. For these schemes the condition of the invariant coupling normalization is of a simple form

$$\bar{g}|_{Q^2=\mu^2} = g_\mu. \quad (6)$$

In a more general case³, that is relevant to minimal subtraction schemes, in particular to the $\overline{\text{MS}}$ one, this condition contains the “normalization function”

$$\bar{g}|_{Q^2=\mu^2} = N(g), \quad (7)$$

³In this section we shall use a general notation g for a coupling constant along with a more specific one $\alpha = g^2/4\pi$ commonly adopted in gauge theories.

that can be dependent on mass(es).

The normalization (6) corresponds to the simplest functional equation (FE) for the invariant coupling, that in the massless case is of the form

$$\bar{g}(x; g_\mu) = \bar{g}\left(\frac{x}{t}; \bar{g}(t; g_\mu)\right) \quad , \quad x = \frac{Q^2}{\mu^2} . \quad (8)$$

Here, the condition (6) follows from the RG functional equation (8). We shall refer to (6) as to the “canonical” normalization condition. Meanwhile, for the case (7), the FE is of a more involved form — see below eq.(14) depending on the mass argument m as well.

3.1 Massive case

Now, we consider a more general case that takes into account an important generalization: the \bar{g} dependence of mass(es) of particle(s). Here, an invariant coupling acquires one more argument that can be chosen (see, *e.g.*, ref.[4]) as a ratio $m^2/\mu^2 = y$, and the normalization condition for an invariant coupling

$$\bar{g}_N(1, y; g) = N(y, g) \quad (9)$$

generally differs from the simplest, “canonical”, form (6). It contains a function N depending on two arguments.

Note here that we have introduced a new special notation \bar{g}_N for the coupling function with the general normalization condition (9) to distinguish it from a coupling function normalized to g . In what follows, we use the term “invariant coupling” (IC) for the function \bar{g}_N with the nontrivial normalization function $N \neq g$ and the term “effective coupling” (EC) for the function \bar{g} with (6).

The effective massive coupling satisfies[1] a rather simple functional

$$\bar{g}(x, y; g) = \bar{g}\left(\frac{x}{t}, \frac{y}{t}, \bar{g}(t, y; g)\right) ; \quad x = \frac{Q^2}{\mu^2}, \quad y = \frac{m^2}{\mu^2}, \quad (10)$$

and differential equations

$$x \frac{\partial \bar{g}(x, y; g)}{\partial x} = \beta\left(\frac{x}{y}, \bar{g}\right), \quad \mathcal{X}_\beta \cdot \bar{g}(x, y; g) = 0 \quad (11)$$

with

$$\mathcal{X}_\beta \equiv \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \beta(y, g) \frac{\partial}{\partial g} \right\} ; \quad \beta(y, g) \equiv \left. \frac{\partial \bar{g}(\xi, y; g)}{\partial \xi} \right|_{\xi=1}. \quad (12)$$

The last two equations (11) and (12) are generalizations of massless equations, and turn to the latter in the limit $y \rightarrow 0$. We define a special notation \mathcal{X}_β for the particular infinitesimal (Lie) group operator (that in the given context was first introduced by Ovsyannikov[18] in 1956). Its superscript refers to the coordinate of the last term. In what follows we shall refer to \mathcal{X} as to the *Lie–Ovsyannikov (LO) operator*.

The effective coupling \bar{g} by definition satisfies the canonical normalization condition

$$\bar{g}(1, y; g) = g, \quad (13)$$

while other RG one-argument functions, like propagator amplitudes, at $Q^2 = \mu^2$ are normalized to unity.

At the same time, the functional equation for the IC is of a more involved form

$$\bar{g}_N(x, y; g) = \bar{g}_N \left\{ \frac{x}{t}, \frac{y}{t}; n \left[\frac{y}{t}, \bar{g}_N(t, y; g) \right] \right\} \quad (14)$$

that reflects the group composition property and corresponds to the normalization condition (9). Here, n is the function reverse to N with respect to the second argument.

By differentiating (14) in two various ways one obtains the differential equations

$$\frac{\partial \bar{g}_N(x, y; g)}{\partial \ln x} = B \left(\frac{y}{x}, \bar{g}_N(x, y; g) \right), \quad \mathcal{X}_\beta \cdot \bar{g}_N(x, y; g) = 0 \quad (15)$$

with

$$B(y, g) = \frac{\partial}{\partial \xi} \bar{g}_N \{ \xi, y; n(y, g) \} \Big|_{\xi=1}; \quad \beta(y, g) = \left\{ B(y, \gamma) \frac{\partial}{\partial \gamma} - y \frac{\partial}{\partial y} \right\} n(y, \gamma) \Big|_{\gamma=N(y, g)}. \quad (16)$$

Note here that for IC we have two *different* beta-functions B and β entering into eqs.(15) and related by the second of equations (16).

Meanwhile, it turns out to be possible to treat invariant coupling \bar{g}_N as an effective coupling function corresponding to some other coupling constant $\gamma \equiv N(y, g)$. Indeed, by introducing a new coupling function

$$\bar{\gamma}(x, y; \gamma) = \bar{g}_N \{ x, y; n(y, \gamma) \}, \quad (17)$$

we find that it satisfies the group functional equation of a simple type (10), the differential eqs. analogous to (11) with the LO operator \mathcal{X}_B with the same generator $B(y, \gamma)$ and can be considered as EC corresponding to the coupling constant γ .

On the other hand, if we define a new auxiliary function \tilde{g} by the relation

$$\tilde{g}(x, y; g) = n \left\{ \frac{y}{x}, \bar{g}_N(x, y; g) \right\} = n \left\{ \frac{y}{x}, \bar{\gamma}(x, y; N(y, g)) \right\}, \quad (18)$$

it will also satisfy the simple functional equation (10) and can be treated on the equal footing with EC $\bar{\gamma}$. Both group differential eqs. for \tilde{g} contain the same generator β . We shall refer to it as to the “group effective coupling” to stress that, generally, it is not related to IC by the Dyson transformation.

3.2 Mass dependent transformations

Consider now, in the general mass dependent case, the transformation from the coupling constant γ and IC $\bar{\gamma}$ to the new coupling constant g and EC \bar{g} . The transition from

popular massless $\overline{\text{MS}}$ to the above-mentioned massive $\overline{\text{MS}}$ can be represented as a sequence of such transformations. Let, without loss of generality, coupling constants g and γ be connected by $\gamma = N(y, g)$, $g = n(y, \gamma)$. Then, according to eq.(18), the effective couplings will be related by

$$\bar{\gamma}(x, y; \gamma) = N\{y/x, \bar{g}(x, y; g)\} \quad (19)$$

and the corresponding group generators by (16).

In other words, it is possible to consider the transition

$$g \rightarrow \gamma = N(y, g) \quad (20)$$

as a coupling constant transformation related to the change of the renormalization prescription. That is, starting with the set $\{g, \bar{g}(x, y; g), \beta(y, g)\}$ we can move to another one $\{\gamma, \bar{\gamma}(x, y; \gamma), B(y, \gamma)\}$ by the transformations (20) and also by (19) represented in the form

$$\bar{g} \rightarrow \bar{\gamma}(x, y; \gamma) = N\left\{\frac{y}{x}, \bar{g}\left[x, y; n\left(\frac{y}{x}, \gamma\right)\right]\right\} \quad (21)$$

that is a composition of (17) and (18) with the generators β and B related by Eq.(16).

Correspondingly, in the group FEq., for an RG covariant function $s(x, y; g)$ (like, *e.g.*, propagator amplitude) it is necessary not only to take into account relation (18) between IC \bar{g}_N and EC \tilde{g}

$$s(x, y; g) = \frac{s(t, y; g)}{s(1, y, \tilde{g}(t, y; g))} \cdot s\left(\frac{x}{t}, \frac{y}{t}; \tilde{g}(t, y; g)\right) \quad (22)$$

but also normalization of s itself

$$s(1, y; g) = S(y, g) \neq 1.$$

The related differential equations are of the form

$$\frac{\partial \ln s(x, y; g)}{\partial \ln x} = \gamma\left[\frac{y}{x}, \tilde{g}(x, y, g)\right]; \quad \mathcal{X}_\beta \cdot \ln s(x, y; g) = \tilde{\gamma}\left(\frac{y}{x}, g\right) \quad (23)$$

with

$$\gamma(y, g) = \frac{\partial}{\partial t} \ln s(t, y; g) \Big|_{t=1} \quad \text{and} \quad \tilde{\gamma}(y, g) = \gamma(y, g) + \left\{ y \frac{\partial}{\partial y} - \beta(y, g) \frac{\partial}{\partial g} \right\} \ln S(y, g). \quad (24)$$

In a given gauge QFT model with one coupling, both EC and IC, besides the renormalization scheme, are specified by the choice of the vertex and the way of its subtraction.

3.3 The “vertex” dependence

In the gauge theories for the mass dependent case in the definition of effective coupling there appears a new specific degree of freedom related to the existence of several Lagrangian structures with the same coupling constant. Here, in defining IC, different vertices can be used. This means that in such a case we have various possibilities for defining IC *in the same RS*. The transition between diverse ICs is described by transformations just considered.

To illustrate, take, e.g., a few dressed QCD vertices, the 3-gluons Γ_{3gl} , the gluon-ghost Γ_{gl-gh} and the gluon-quark Γ_{gl-q} ones.

Generally, each of them could be used for the QCD invariant coupling defining, *e.g.*,

$$\bar{\alpha}_{3gl} = \alpha_s \Gamma_{3gl}^2 Z^3, \quad \bar{\alpha}_{gl-gh} = \alpha_s \Gamma_{gl-gh}^2 Z G^2, \quad \bar{\alpha}_{gl-q} = \alpha_s \Gamma_{gl-q}^2 Z S^2 \quad (25)$$

with Z , G and S — gluon, ghost and quark propagator scalar amplitudes.

In the UV massless limit, all definitions are, in a sense, equivalent: they coincide in the $\overline{\text{MS}}$ scheme and could be “slightly different” in MOM schemes due to various definitions of the “reduced” vertex functions presented as functions of one space-like argument Q^2 .

However, in the massive case they could be drastically different in the IR region due to diverse dependence on the light quark masses. As it is well known, the quark (fermionic) propagator as well as the gluon-quark vertex are singular on the quark mass shell. For the light quarks this leads to the singular IR behavior.

The gluon-ghost vertex and related coupling $\bar{\alpha}_{gl-gh}$ seems to be less sensitive to the mass effects. Indeed, at the one-loop level, quark mass effects can contribute to it only via the gluon propagator factor Z . Meanwhile, the polarization loop is free of afore-mentioned singularity. Two others ICs, $\bar{\alpha}_{3gl}$ and $\bar{\alpha}_{gl-q}$ are more sensitive. In each of them, besides three propagator amplitudes, the mass effects come also via the vertices Γ_{3gl} or Γ_{gl-q} .

Quite probably, just to this there corresponds a more “quiet” variant of the IR behavior for $\bar{\alpha}(Q^2)$ obtained by the Tübingen group.

4 The model coupling transformations

Here, we present some model transformations of the coupling constant and related IC transformations, including the mass-dependent ones. It is shown that an appropriate α_s transformation could drastically change the properties of the corresponding running coupling as a function of Q^2 in the IR region.

4.1 Examples from APT of massless transformations

Let us start with examples induced by the *Analytic Perturbation Theory* (APT) that has been devised recently [19, 20] to clean out the perturbative QCD (in a “bloodless” way — by imposing the Källen-Lehmann analyticity) of unphysical singularities like the Landau ghost pole.

In the APT, the transition from the usual invariant $\overline{\text{MS}}$ coupling constant α_s to the Minkowskian α_M and Euclidean α_E ones can be treated [21] as a coupling transformations similar to a change of the renormalization scheme (RS). At the one-loop case

$$\alpha_s \rightarrow \alpha_M(\alpha_s) = \frac{1}{\pi\beta_0} \arccos \frac{1}{\sqrt{1 + \pi^2\beta_0^2\alpha_s^2}} = \frac{1}{\pi\beta_0} \arctan(\pi\beta_0\alpha_s), \quad (26)$$

$$\alpha_s \rightarrow \alpha_E(\alpha_s) = \alpha_s + \frac{1}{\beta_0} (1 - e^{1/\beta_0\alpha_s})^{-1}, \quad (27)$$

Here, the first transition “looks quite usual”⁴ as α_M can be expanded in powers of α_s (like in the usual RS transformation), while the second one in the weak coupling case is close to the identity transformation as far as the nonperturbative term $\simeq e^{-1/\beta_0 \alpha_s}$ leaves no “footsteps” in the power expansion.

If one starts with the common one-loop $\bar{\alpha}_s(Q^2) \sim 1/\ln(Q^2/\Lambda^2)$ with its first order pole at $Q^2 = \Lambda^2$, then, as a result of substitution of (26) and (27) into the RG differential equation and its integration, one arrives at the ghost-free expressions

$$\tilde{\alpha}(Q^2) = \frac{1}{\pi\beta_0} \arccos \frac{L}{\sqrt{L^2 + \pi^2}} \Big|_{L>0} = \frac{\arctan(\pi/L)}{\pi\beta_0}; \quad L = \ln \frac{Q^2}{\Lambda^2}, \quad (28)$$

$$\alpha_{\text{an}}(Q^2) = \frac{1}{\beta_0} \left[\frac{1}{L} - \frac{\Lambda^2}{Q^2 - \Lambda^2} \right] \quad (29)$$

that are well known in the APT.

The transformation functions $\alpha_M(\alpha_s)$ and $\alpha_E(\alpha_s)$ obey important properties. They tend to α_s in the weak coupling limit

$$\alpha_i \rightarrow \alpha_s \quad \text{at} \quad \alpha_s \ll 1 \quad [\text{AF}]$$

and

$$\underline{\text{are finite at } \alpha_s = \infty.} \quad [\text{GhF}]$$

Besides, they are finite

$$\alpha_i \rightarrow 1/\beta_0 \quad \text{in the limit} \quad \alpha_s \rightarrow -0. \quad [\text{IRf}]$$

The first property [AF] provides a correspondence with the weak coupling limit with its asymptotic freedom property. The second one [GhF] reflects the absence of ghosts and the [IRf] one relates to the finite IR limit.

This last feature is absent in the transformation

$$\alpha_s \rightarrow \alpha_N = \alpha_s (1 - e^{-1/\beta_0 \alpha_s}) \quad (30)$$

that is equivalent to the one obtained [23] in the modified APT.

Just due to the [AF], [GhF] and [IRf] properties, both APT ICs (28) and (29)⁵

- (x) enjoy the Asymptotic Freedom,
- (xx) are free of ghost singularities at $Q^2 = \Lambda^2$;
- (xxx) have a finite IR limit at $Q^2 = 0$.

⁴The same function of the initial coupling constant $F(g) \sim \arctan(g)$ appeared [22] in the exact solution of the two-dimensional Thirring model.

⁵For both $\tilde{\alpha}$ and α_{an} the corresponding beta-functions have zero at $\alpha = 1/\beta_0$ and are symmetric under the reflection $[\alpha - 1/2\beta_0] \rightarrow -[\alpha - 1/2\beta_0]$. Meanwhile, the beta-function for $\tilde{\alpha}(s)$ turns out to be equal to the spectral function for $\alpha_{\text{an}}(Q^2)$. This last property, that provides a peculiar realization of the Schwinger hypothesis[24], is valid [21, 25] outside the one-loop approximation.

At the same time, the last transformation (30) satisfying only [AF] and [Ghf], yields an expression obeying the singular IR behaviour

$$\bar{\alpha}_N(Q^2) = \frac{Q^2 - \Lambda^2}{\beta_0 Q^2 \ln(Q^2/\Lambda^2)}. \quad (31)$$

It has an “extra Q^{-2} factor” that, as some people believe, relates to the linear growth of the interquark potential.

Consider one more massless transformation

$$\alpha_s \rightarrow \alpha_{SF} = \alpha_s \frac{e^b - \exp\{be^{-1/2\beta_0 \alpha_s}\}}{e^b - 1}; \quad b = \frac{m}{\Lambda}, \quad (32)$$

leading to the expression

$$\bar{\alpha}_{SF}(Q^2) = \frac{1}{\beta_0 \ln(\frac{Q^2}{\Lambda^2})} \cdot \frac{e^b - e^{b\frac{\Lambda}{Q}}}{e^b - 1} \quad (33)$$

with the exponential singularity $\alpha_{SF} \sim e^{M/Q}$ as $Q \rightarrow 0$. It reminds an analytic approximation (5) of recent ALPHA group numerical results transposed (see, *e.g.*, Fig.3 in Ref.[17]) to the IR momentum region with the help of “quantum–mechanical correspondence” $L \rightarrow 1/Q$. Such an essential singularity contradicts the Källén–Lehmann representation. Note also that expression (32), in turn, obeys an essential singularity at $\alpha_s \rightarrow +0$.

From the given examples, there follows a simple “rule of correspondence” between transformations (26) — (32) and resulting expressions (28) — (33) for invariant couplings:

under the coupling constant transition $\alpha_s \rightarrow \alpha_i = f_i(\alpha_s)$ the invariant coupling function transformation is

$$\bar{\alpha}_s(Q^2) \rightarrow \bar{\alpha}_i(Q^2) = f_i(\bar{\alpha}_s(Q^2)) \quad (34)$$

4.2 Massive transformation

Consider now the mass–dependent transformation of a coupling constant. Generally, according to (20), it involves a function of two variables

$$\alpha_s \rightarrow \alpha^* = N(y, \alpha_s). \quad (35)$$

Here, the IC transformation looks like

$$\bar{\alpha}_s \rightarrow \bar{\alpha}^*(x, y; \alpha^*) = N\left\{\frac{y}{x}, \bar{\alpha}_s\left[x, y; n\left(\frac{y}{x}, \alpha_s\right)\right]\right\}. \quad (36)$$

The r.h.s. of the last relation, in distinction to (34), contains the variable $y/x = m^2/Q^2$ that can influence the IR behavior.

To illustrate, take a model expression

$$N(y, \alpha) = \alpha_M(\alpha_s) \left(\frac{1-y}{1+y} \right)^{c \alpha_M(\alpha_s)}, \quad (37)$$

that yields

$$\bar{\alpha}^*(Q^2) = \tilde{\alpha}(Q^2) \left(\frac{Q^2 - m^2}{Q^2 + m^2} \right)^{c \tilde{\alpha}(Q^2)}. \quad (38)$$

Here, the coupling constant $\alpha_M(\alpha_s)$ and EC $\tilde{\alpha}(Q^2)$ are defined by expressions of the type (26) and (28), or by their more involved two- or three-loop counterparts — see, *e.g.*, [21, 25]. Here, it is essential that $\tilde{\alpha}(Q^2)$ is free of unphysical singularities and monotonically increases to a finite IR limit $\tilde{\alpha}(0) \sim 1$. The structure of mass singularity in (38) resembles the well-known IR one (see, [4]) of the QED vertex on the mass shell.

For positive $c \simeq 1$ and small quark mass $m \ll \Lambda$ values the function $\bar{\alpha}^*(Q^2)$, defined by (38), can be very close to the numerical results for $\bar{\alpha}_s$, obtained by the Paris group (see subsection 2.2) and by the “Transoceanic team” (subsection 2.3).

5 Discussion

As it has been mentioned above in Section 2, the results of various groups for the QCD invariant coupling in the momentum representation — obtained by lattice simulations of the Euclidean functional integral as well as by approximate solving of truncated SDEs — turn out to be quite different in the IR region. At the same time, their “physical” results for hadronic properties of matter seem to be more correlated.

Our “model constructions” of Section 4 demonstrate that the IR properties essentially depend on a precise way of the IC defining. We used a class of ICs that in the UV region correlates with the perturbative QCD $\bar{\alpha}_s$ coupling in the $\overline{\text{MS}}$ scheme. Just this correspondence is usually considered to be essential in lattice simulations and analysis of SDEs solutions.

In practice, these “admissible” ICs, satisfying the condition [AF], can correspond to diverse lattice calculations. As far as these calculations satisfactorily describe confinement and hadronic physics, it is reasonable to consider them as “physical” ones.

Special attention should be paid to EC $\bar{g}^2(L)$ of the ALPHA group with its exponential growth $\sim e^{ML}$ with lattice spatial size L . To interpret this result in the momentum transfer representation, one needs to be very cautious with performing the Fourier transformation as far as the usual Tauber criterion is not valid. This issue will be considered in more detail elsewhere.

To conclude, we believe that there is no direct physical sense in attempts to establish some “correct IR behavior” of the perturbative QCD invariant coupling. Any “infrared QCD physics” like hadronic and τ decay ones, cannot be described in terms of only pQCD notions. Either pQCD description has to be supplemented by some additional semiphenomenological parameters like “effective parton masses” [26, 27, 28] (of order of pion mass) and anomalous vacuum averages, or some other, intrinsically nonperturbative, means, like lattice simulations or the Dyson–Schwinger equations, should be used.

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